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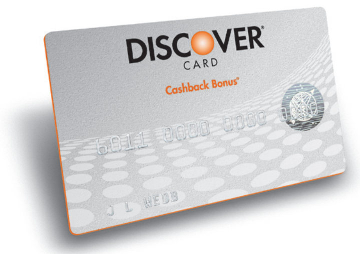
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7 Complex Numbers and Polar Coordinates

In this chapter we examine more applications of the trigonometric functions. We first define complex numbers, and then show how trigonometry is useful in representing these numbers as well as computing with them.

We then introduce polar coordinates, a system of coordinates that is useful in describing many situations where periodic motion is involved.

7–1 Complex numbers

Complex numbers were developed hundreds of years ago, but they became much more important about 100 years ago when they began to be used to model physical phenomena, particularly in the study of electricity. Their development and use centers on the number *i*.

The *i* stands for the word *imaginary;* and represents $\sqrt{-1}$. We learned in the past that the square root of a negative number is not defined since, for example, if $i = \sqrt{-1}$, then $i^2 = -1$, and we know that the square of any real number is positive or zero. Nevertheless, equations like $x^2 = -1$ produce the "solution" $\sqrt{-1}$. To the mathematicians of the sixteenth and later centuries these numbers were bothersome. They keep occurring when solving equations, yet, to these people, such numbers could not exist. René Descartes coined the term "imaginary" to describe solutions to equations that involved the square roots of negative numbers, and this term has persisted to this day. (In fact, the term "real" number was coined in reaction to the word "imaginary.")

It turned out, however, that the study of certain physical laws almost demanded the existence of $\sqrt{-1}$, and today the existence of a number such as i is no longer doubted. Since the property $i^2 = -1$ is incompatible with the properties of the real number system, a new number system was introduced to incorporate these imaginary numbers. This system is the complex number system, and we now study its properties.

The square root of −1

$$i = \sqrt{-1}$$

Rectangular form of a complex number

A **complex number** is an expression of the form $a + bi$, where a and b are real numbers. a is called the *real part*, and b is called the *imaginary part* of the number. $a + bi$ is called the **rectangular form** of a complex number.

$2 + 3i$ is an example of a complex number; 2 is its real part and 3 is its imaginary part.

Example 7–1 A

Identify the real and imaginary parts of the following complex numbers:

a. $5 + 7i$ **b.** $2 - i$ **c.** 4 **d.** $-6i$

a. Real part: 5; imaginary part: 7.

b. Real part: 2; imaginary part: −1. $2 - i = 2 + (-1)i$

c. If we represent 4 as $4 + 0i$, we see that the real part is 4 and the imaginary part is 0.

d. Represent $-6i$ as $0 - 6i$ to see that the real part is 0 and the imaginary part is −6. A complex number in which the real part is zero is usually called a pure imaginary number.

Complex conjugate

$a - bi$ is called the **complex conjugate** of $a + bi$.

Example 7–1 B

Form the complex conjugate of each complex number.

1. $5 + 7i$

$5 - 7i$ is the complex conjugate of $5 + 7i$.

2. $4 - 5i$

$4 + 5i$ is the complex conjugate of $4 - 5i$.

3. 7

We rewrite 7 as $7 + 0i$, whose complex conjugate is $7 - 0i$ or 7. Thus, 7 is the complex conjugate of 7.

4. $-9i$

We rewrite $-9i$ as $0 - 9i$, whose complex conjugate is $0 + 9i$ or $9i$. Thus, $9i$ is the complex conjugate of $-9i$.

An algebra exists for the complex numbers. Rules have been established for the addition, subtraction, multiplication, and division of complex numbers. These operations obey the rules of the real number system if we treat i as a

symbol having the property $i^2 = -1$. The definitions of these operations are given and illustrated in the following paragraphs. First, however, we must define what it means for two complex numbers to be equal.

Equality of complex numbers

The complex numbers $a + bi$ and $c + di$ are *equal* if and only if $a = c$ and $b = d$.

The complex numbers are not ordered; that is, a given complex number is never said to be greater than or less than another complex number.

We now define and illustrate addition and subtraction.

Addition and subtraction of complex numbers

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

Concept

To add or subtract two complex numbers, add or subtract the real parts and the imaginary parts separately.

■ ***Example 7–1 C***

Perform the indicated operations and simplify.

1. $(5 + 4i) + (3 + 2i)$

$$(5 + 4i) + (3 + 2i) = 5 + 3 + 4i + 2i = 8 + 6i$$

2. $(5 + 4i) - (3 + 2i)$

$$(5 + 4i) - (3 + 2i) = 5 - 3 + 4i - 2i = 2 + 2i$$

■

Multiplication of complex numbers

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Concept

This definition is equivalent to our usual rules for the multiplication of real number expressions if we replace i^2 by -1.

$$\begin{aligned} &(a + bi)(c + di) && \text{Multiply as if real expressions} \\ &= ac + adi + bci + bdi^2 && \\ &= ac + adi + bci - bd && \text{Remember that } i^2 = -1 \\ &= (ac - bd) + (ad + bc)i && \end{aligned}$$

■ ***Example 7–1 D***

Perform the indicated operations and simplify.

1. $(5 + 4i)(3 + 2i)$

$$\begin{aligned} (5 + 4i)(3 + 2i) &= 15 + 10i + 12i + 8i^2 \\ &= 15 + 22i + 8(-1) \\ &= 15 + 22i - 8 \\ &= 7 + 22i \end{aligned}$$

2. $(-2 + 5i)(3 - 5i)$

$$\begin{aligned}(-2 + 5i)(3 - 5i) &= -6 + 10i + 15i - 25i^2 \\ &= -6 + 25i - 25(-1) \\ &= -6 + 25i + 25 \\ &= 19 + 25i\end{aligned}$$

3. $5i(6 - 4i)$

$$\begin{aligned}5i(6 - 4i) &= 30i - 20i^2 \\ &= 30i + 20 \\ &= 20 + 30i\end{aligned}$$

Note We always write the real part of a complex number first.

Division of complex numbers

$$\frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i$$

Concept

The above result is obtained by multiplying the numerator and denominator of the quotient by the conjugate of the denominator.

$$\begin{aligned}\frac{a + bi}{c + di} &= \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} \\ &= \frac{ac - adi + bci - bdi^2}{c^2 - cdi + cdi - d^2i^2} \\ &= \frac{ac + bd + (bc - ad)i}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i\end{aligned}$$

Example 7–1 E

Perform the indicated operations and simplify each expression.

1. $\dfrac{5 + 4i}{3 + 2i}$

$$\begin{aligned}\frac{5 + 4i}{3 + 2i} &= \frac{5 + 4i}{3 + 2i} \cdot \frac{3 - 2i}{3 - 2i} = \frac{15 - 10i + 12i - 8i^2}{9 - 6i + 6i - 4i^2} \\ &= \frac{15 + 2i + 8}{9 + 4} \\ &= \frac{23 + 2i}{13} \\ &= \tfrac{23}{13} + \tfrac{2}{13}i\end{aligned}$$

Note A complex number must have two parts, a real and an imaginary part. This is why we performed the last step in the previous part.

2. $\dfrac{-2 + 5i}{3 - 4i}$

$$\frac{-2 + 5i}{3 - 4i} = \frac{-2 + 5i}{3 - 4i} \cdot \frac{3 + 4i}{3 + 4i} = \frac{-6 - 8i + 15i - 20}{9 + 12i - 12i + 16}$$
$$= \frac{-26 + 7i}{25}$$
$$= -\tfrac{26}{25} + \tfrac{7}{25}i$$

3. $\dfrac{5i}{6 - 4i}$

$$\frac{5i}{6 - 4i} = \frac{5i}{6 - 4i} \cdot \frac{6 + 4i}{6 + 4i} = \frac{30i - 20}{36 + 16} = -\frac{20}{52} + \frac{30}{52}i$$
$$= -\tfrac{5}{13} + \tfrac{15}{26}i$$

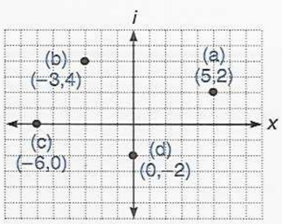


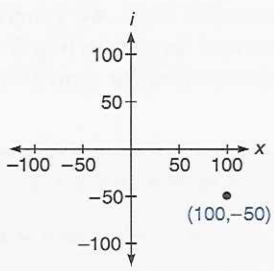
Figure 7–1

Complex numbers can be graphed as ordered pairs in a rectangular coordinate system by letting horizontal distances represent the real part and vertical distances represent the imaginary part of each complex number. Figure 7–1 shows the graph of (a) $5 + 2i$, (b) $-3 + 4i$, (c) -6, and (d) $-2i$. A graph of complex numbers, such as those in figure 7–1, is called an **Argand diagram.** Example 7–1 F illustrates one application.

■ ***Example 7–1 F***

In alternating current theory in electronics, complex numbers are used to represent *impedance Z,* a measure of the way in which a circuit retards the flow of current through it. The real part of an impedance is called the *resistance R,* and the imaginary part is called the *reactance X.* Thus, $Z = R + Xi$. The units for Z, R, and X are ohms. Graph circuit impedance Z if $R = 100$ ohms and $X = -50$ ohms.

$Z = R + Xi = 100 - 50i$. The graph is shown in the figure.



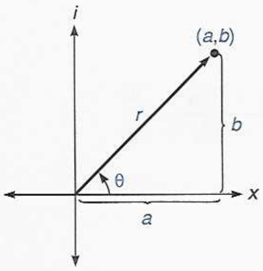


Figure 7–2

Polar form of a complex number

We can use the graph of a complex number as a guide to develop another way to represent a complex number. Given the complex number $a + bi$, let r represent the distance from the origin to the point (a,b) and let θ represent the angle in standard position determined by the ray containing the origin and (a,b). See figure 7–2. Using the definitions of section 2–2 we obtain $\cos\theta = \frac{a}{r}$ and $\sin\theta = \frac{b}{r}$. Thus, $a = r\cos\theta$ and $b = r\sin\theta$. This means we can rewrite $a + bi$ as $r\cos\theta + (r\sin\theta)i$, or $r(\cos\theta + i\sin\theta)$. The expression $\cos\theta + i\sin\theta$ occurs so often it is abbreviated as **cis** θ, so $a + bi = r\text{ cis }\theta$. This form is called the **polar form** of a complex number.[1] Also, note that $r = \sqrt{a^2 + b^2}$ and that $\tan\theta = \frac{b}{a}$.

> **Polar form of a complex number**
>
> If $z = a + bi$ is a complex number that determines an angle θ, then
> $$z = r\text{ cis }\theta$$
> is its polar form, where cis θ means $\cos\theta + i\sin\theta$, and
> $$r = \sqrt{a^2 + b^2}$$
> The value r is called the **modulus** of z, which is also written $|z|$.

Note 1. The value of θ is not unique. All coterminal values produce the same rectangular form. Thus, 2 cis 10° is equivalent to 2 cis 370°.
2. The modulus is the distance from the origin to the complex coordinate (a,b).

Polar-rectangular conversions

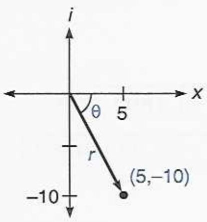
As with vectors (section 6–3), we generally give a value of θ so that $-180° < \theta \le 180°$. A method for converting from rectangular form to polar form is a paraphrase of the method for converting vectors from rectangular to polar form.

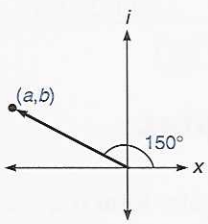
> Given a complex number $z = a + bi = r\text{ cis }\theta$. Then,
> $$r = \sqrt{a^2 + b^2}, \quad \tan\theta = \tan^{-1}\frac{b}{a} \text{ (if } a \neq 0\text{)}, \quad \text{and}$$
> $$\theta = \begin{cases} \theta' & \text{if } a > 0 \\ \theta' - 180° \text{ if } \theta' > 0 & \text{if } a < 0 \\ \theta' + 180° \text{ if } \theta' < 0 & \end{cases}$$

Note If $a = 0$ then θ is 90° if $b > 0$, and $-90°$ if $b < 0$. A sketch will make the choice clear.

[1]In 1893 Irving Stringham first used the notation cis $\beta = \cos\beta + i\sin\beta$.

■ ***Example 7–1 G***





Convert between polar and rectangular form.

1. $5 - 10i$

The graph is shown in the figure.

$$r = |5 - 10i| = \sqrt{5^2 + (-10)^2} = \sqrt{125} = 5\sqrt{5}$$

$$\theta' = \tan^{-1}\frac{b}{a} = \tan^{-1}\left(\frac{-10}{5}\right) = \tan^{-1}(-2), \text{ so } \theta' \approx -63.4^\circ$$

$$\theta = \theta' \approx -63.4^\circ \quad a > 0$$

Thus, the polar form of $5 - 10i$ is $5\sqrt{5}\ \text{cis}(-63.4^\circ)$ or about $(11.1,-63.4^\circ)$.

2. $5 \text{ cis } 150^\circ$

$$\begin{aligned} 5 \text{ cis } 150^\circ &= 5(\cos 150^\circ + i \sin 150^\circ) \\ &= 5\left(-\frac{\sqrt{3}}{2} + i \cdot \frac{1}{2}\right) \\ &= -\frac{5\sqrt{3}}{2} + \frac{5}{2}i \end{aligned}$$

Thus, $5 \text{ cis } 150^\circ = -\dfrac{5\sqrt{3}}{2} + \dfrac{5}{2}i$ or about $-4.3 + 2.5i$. ■

Polar-rectangular conversions on a calculator

As stated in the previous section, most calculators are programmed to perform polar/rectangular conversions. This conversion works equally well for vectors (section 6–3) and for complex numbers. The conversion is done with keys marked "R → P" or "→ P" and "P → R" or "→ R" or something equivalent. The results are stored in locations referred to as x and y. Typical keystrokes are illustrated here. (The TI-81 steps are shown below.)

Part 1 of example 7–1 G would be done as follows.

$z = 5 - 10i$

5 $\boxed{\text{R} \to \text{P}}$ 10 $\boxed{+/-}$ $\boxed{=}$ Display: $\boxed{11.18033989}$

$\boxed{x \leftrightarrow y}$ Display: $\boxed{-63.43494882}$

Thus, $z \approx (11.1,-63.4^\circ)$.

Part 2 of example 7–1 G would be done in the following way.

$z = 5 \text{ cis } 150^\circ$

5 $\boxed{\text{P} \to \text{R}}$ 150 $\boxed{=}$ Display: $\boxed{-4.330127019}$

$\boxed{x \leftrightarrow y}$ Display: $\boxed{2.5}$

Thus, $z \approx -4.3 + 2.5i$.

The TI-81 uses the values X, Y, R, and θ. (Y, R, θ are $\boxed{\text{ALPHA}}$ 1, $\boxed{\times}$, and 3, respectively.) It also uses the two MATH functions "R ▸ P(" (rectangular to polar) and "P ▸ R(" (polar to rectangular).

$\boxed{\text{MODE}}$ Deg $\boxed{\text{ENTER}}$ Make sure the calculator is in degree mode.

Example 7–1 G, part 1:

$\boxed{\text{MATH}}$ 1 5 $\boxed{\text{ALPHA}}$ $\boxed{.}$ $\boxed{(-)}$ 10 $\boxed{)}$

$\boxed{\text{ENTER}}$ Display: $\boxed{11.18033989}$

$\boxed{\text{ALPHA}}$ 3 $\boxed{\text{ENTER}}$ Display: $\boxed{-63.43494882}$

Example 7–1 G, part 2:

$\boxed{\text{MATH}}$ 2 5 $\boxed{\text{ALPHA}}$ $\boxed{.}$ 150 $\boxed{)}$

$\boxed{\text{ENTER}}$ Display: $\boxed{-4.330127019}$

$\boxed{\text{ALPHA}}$ 1 $\boxed{\text{ENTER}}$ Display: $\boxed{2.5}$

Multiplication and division of complex numbers in polar form

Multiplication and division of complex numbers in rectangular form is quite complicated. The following theorems show that these procedures are quite simple when the complex numbers are in polar form.

Complex multiplication—polar form

$$(r_1 \text{ cis } \theta_1)(r_2 \text{ cis } \theta_2) = r_1r_2 \text{ cis } (\theta_1 + \theta_2)$$

Complex division—polar form

$$\frac{r_1 \text{ cis } \theta_1}{r_2 \text{ cis } \theta_2} = \frac{r_1}{r_2} \text{ cis } (\theta_1 - \theta_2),\ r_2 \neq 0$$

Concept

To *multiply,* multiply the moduli and add the angles. To *divide,* divide the moduli and subtract the angles.

We can see that the first theorem is true by converting the complex numbers into rectangular form and performing the multiplication as defined earlier in this section.

$(r_1\text{cis}\theta_1)(r_2\text{cis}\theta_2)$

$= (r_1\cos\theta_1 + ir_1\sin\theta_1)(r_2\cos\theta_2 + ir_2\sin\theta_2)$

$= r_1r_2\cos\theta_1\cos\theta_2 + ir_1r_2\cos\theta_1\sin\theta_2 + ir_1r_2\sin\theta_1\cos\theta_2 + i^2r_1r_2\sin\theta_1\sin\theta_2$

$= r_1r_2\cos\theta_1\cos\theta_2 - r_1r_2\sin\theta_1\sin\theta_2 + ir_1r_2\cos\theta_1\sin\theta_2 + ir_1r_2\sin\theta_1\cos\theta_2$

Recall that $i^2 = -1$

$= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$

Now use the identities for $\cos(\alpha + \beta)$ and $\sin(\alpha + \beta)$ from section 5–2.

$= r_1r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$

$= r_1r_2\text{cis}(\theta_1 + \theta_2)$

The proof of the process for division in polar form is left as an exercise.

In electronics, Ohm's law states $V = IZ$, where V means voltage, I means current, and Z means impedance. The units are volts, amperes, and ohms, respectively. Often complex numbers are used to describe the values of volts, amperes, and ohms. The fact that the angles add in forming the product models the physical situation in an electronics circuit. This use is illustrated in example 7–1 H, part 2.

■ *Example 7–1 H*

Multiply or divide the complex numbers.

1. $(2 \text{ cis } 110°)(10 \text{ cis } 300°)$

$$\begin{aligned} &= 2 \cdot 10 \text{ cis}(110° + 300°) \\ &= 20 \text{ cis } 410° \\ &= 20 \text{ cis } 50° \qquad \text{410° and 50° are coterminal} \end{aligned}$$

2. In a certain electronic circuit current $I = 4 \text{ cis } 30°$ amperes and impedance $Z = 2 \text{ cis } 15°$. Use Ohm's law, $V = IZ$, to compute voltage V.

$$\begin{aligned} V &= IZ \\ &= (4 \text{ cis } 30°)(2 \text{ cis } 15°) \\ &= 8 \text{ cis } 45° \text{ (volts)} \end{aligned}$$

3. $\dfrac{15 \text{ cis } 30°}{18 \text{ cis } 80°}$

$$= \tfrac{15}{18}\text{cis}(30° - 80°) = \tfrac{5}{6}\text{cis}(-50°)$$ ■

De Moivre's theorem

Consider the following computations for successive powers of a complex number $r \text{ cis } \theta$.

$$\begin{aligned} (r \text{ cis } \theta)^1 &= r \text{ cis } \theta \\ (r \text{ cis } \theta)^2 &= (r \text{ cis } \theta)(r \text{ cis } \theta) = r^2 \text{ cis } 2\theta \\ (r \text{ cis } \theta)^3 &= (r \text{ cis } \theta)(r \text{ cis } \theta)^2 = (r \text{ cis } \theta)(r^2 \text{ cis } 2\theta) = r^3 \text{ cis } 3\theta \\ (r \text{ cis } \theta)^4 &= (r \text{ cis } \theta)(r \text{ cis } \theta)^3 = (r \text{ cis } \theta)(r^3 \text{ cis } 3\theta) = r^4 \text{ cis } 4\theta \end{aligned}$$

It is logical to assume that the pattern above continues. This is true, and the result is called **De Moivre's theorem.** It actually turns out that the exponent can be any real number.

De Moivre's theorem

$(r \text{ cis } \theta)^n = r^n\text{cis } n\theta$ for any real number n.

This theorem is illustrated in example 7–1 I.

■ *Example 7–1 I*

Use De Moivre's theorem to compute the following.

1. $(5 \text{ cis } 137°)^3$; leave the answer in polar form.

$$\begin{aligned} &= 5^3 \text{ cis}(3 \cdot 137°) \\ &= 125 \text{ cis } 411° \\ &= 125 \text{ cis } 51° \end{aligned}$$

2. $(1 + 0.8i)^6$; leave the answer in rectangular form; round to the nearest tenth.

We could of course multiply $(1 + 0.8i)$ by itself five times to obtain the result. The amount of work is prohibitive. If we put the number in polar form we can use De Moivre's theorem.

$$|1 + 0.8i| = \sqrt{1^2 + 0.8^2} \approx 1.281;\ \theta = \tan^{-1}\frac{0.8}{1} \approx 38.66°$$

Then $1 + 0.8i \approx 1.281 \text{ cis } 38.66°$. Then

$$\begin{aligned}(1 + 0.8i)^6 &\approx (1.281 \text{ cis } 38.66°)^6 \\ &= 1.281^6 \text{ cis}(6 \cdot 38.66°) \\ &\approx 4.42 \text{ cis } 231.96° \\ &= 4.42 \cos 231.96° + 4.42 \sin 231.96° i \\ &\approx -2.7 - 3.5i\end{aligned}$$

Thus, $(1 + 0.8i)^6 \approx -2.7 - 3.5i$, to the nearest tenth.

De Moivre's theorem for roots

In the complex number system, every number except 0 has n nth roots; that is, two square roots, three cube roots, four fourth roots, etc. These can be expressed by De Moivre's theorem by replacing n by $\frac{1}{n}$; recall that $x^{\frac{1}{2}} = \sqrt{x}$, $x^{\frac{1}{3}} = \sqrt[3]{x}$, $x^{\frac{1}{4}} = \sqrt[4]{x}$, etc.

> **De Moivre's theorem for roots**
> The n nth roots of $r \text{ cis } \theta$ are of the form
> $$r^{\frac{1}{n}} \text{ cis}\left(\frac{\theta}{n} + \frac{k \cdot 360°}{n}\right),\ 0 \le k < n$$
> where k and n are positive integers.

We can show that any number of the form above is an nth root of $r \text{ cis } \theta$ by raising it to the nth power.

$$\begin{aligned}\left[r^{\frac{1}{n}} \text{cis}\left(\frac{\theta}{n} + \frac{k \cdot 360°}{n}\right)\right]^n &= \left(r^{\frac{1}{n}}\right)^n \text{cis}\left[n\left(\frac{\theta}{n} + \frac{k \cdot 360°}{n}\right)\right] \\ &= r \text{ cis}(\theta + k \cdot 360°) \\ &= r \text{ cis } \theta \qquad \theta + k \cdot 360° \text{ is coterminal to } \theta\end{aligned}$$

If $k \ge n$ we get a repetition of a previous root. The proof of this is left as an exercise. The proof of the fact that the roots are all distinct and that there are no other roots is beyond the scope of this text.

■ ***Example 7–1 J***

Find the roots.

1. Find the three cube roots of 1.

$$1 = 1 \text{ cis } 0°$$

Evaluate $1^{\frac{1}{3}} \text{ cis}\left(\frac{0°}{3} + \frac{k \cdot 360°}{3}\right) = \text{cis}(k \cdot 120°)$ for $k = 0, 1, 2$.

$k = 0$: $\text{cis } 0° = \cos 0° + i \sin 0° = 1 + 0i = 1$

$k = 1$: $\text{cis } 120° = \cos 120° + i \sin 120° = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$k = 2$: $\text{cis } 240° = \cos 240° + i \sin 240° = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

Thus, the three cube roots of 1 are $1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

2. Find decimal approximations (to tenths) of the four fourth roots of $10 - 2\sqrt{39}i$.

$$10 - 2\sqrt{39}i \approx 16 \text{ cis } 308.68°$$

$$16^{\frac{1}{4}} \text{ cis}\left(\frac{308.68°}{4} + \frac{k \cdot 360°}{4}\right) \approx 2 \text{ cis}(77.17 + k \cdot 90°)$$

Evaluate $2 \text{ cis}(77.17 + k \cdot 90°)$ for $k = 0, 1, 2, 3$.

$k = 0$: $2 \text{ cis } 77.17° = 2(\cos 77.17° + i \sin 77.17°) = 0.4 + 2.0i$
$k = 1$: $2 \text{ cis } 167.17° = 2(\cos 167.17° + i \sin 167.17°) = -2.0 + 0.4i$
$k = 2$: $2 \text{ cis } 257.17° = 2(\cos 257.17° + i \sin 257.17°) = -0.4 - 2.0i$
$k = 3$: $2 \text{ cis } 347.17° = 2(\cos 347.17° + i \sin 347.17°) = 2.0 - 0.4i$

Thus, the four fourth roots of $10 - 2\sqrt{39}i$ are approximately $0.4 + 2.0i$, $-2.0 + 0.4i$, $-0.4 - 2.0i$, and $2.0 - 0.4i$.

3. In electronics apparent power P can be determined by $P = I^2Z$. If we solve this for current I we get $I = \pm\sqrt{\frac{P}{Z}}$. Use $I = \sqrt{\frac{P}{Z}}$ to determine I to the nearest tenth if $P = 10 - 2i$ and $Z = 1 + 3i$.

$$P = 10 - 2i \approx 10.2 \text{ cis } 348.69°$$

$$Z = 1 + 3i \approx 3.16 \text{ cis } 71.57°$$

$$\frac{P}{Z} \approx \frac{10.2 \text{ cis } 348.69°}{3.16 \text{ cis } 71.57°} \approx 3.23 \text{ cis } 277.12°$$

We evaluate the expression $3.23^{\frac{1}{2}} \text{ cis}\left(\frac{277.12°}{2} + \frac{k \cdot 360°}{2}\right)$ for $k = 0, 1$.

$k = 0$: $\sqrt{3.23} \text{ cis } 138.56° \approx -1.3 + 1.2i$
$k = 1$: $\sqrt{3.23} \text{ cis } 318.56° \approx 1.3 - 1.2i$

Thus, the current I is about $-1.3 + 1.2i$ or $1.3 - 1.2i$ (amperes). ■

Mastery points

Can you

- Identify the real and imaginary parts of a complex number?
- Form the conjugate of a complex number?
- Evaluate complex expressions involving addition, subtraction, multiplication, and division?
- Graph a complex number when given in rectangular or polar form?
- Convert between the rectangular and polar forms of complex numbers?
- Multiply and divide complex numbers in polar form?
- State and use De Moivre's theorem for integral powers?
- State and use De Moivre's theorem for roots?

Exercise 7–1

Identify the real and imaginary parts of the following complex numbers.

1. $4 - 5i$
2. $3 + 11i$
3. $-4 + i$
4. $3i$
5. 12
6. $-i$
7. $-10 + 2i$
8. $-9 - i$

For each problem (a) find the complex conjugate and (b) graph the number and its conjugate in the same graph.

9. $4 - 5i$
10. $3 + 11i$
11. $-4 + i$
12. $3i$
13. 12
14. $-i$
15. $-10 + 2i$
16. $-9 - i$

Simplify each expression by performing the indicated operations. Do all operations in rectangular form (do not convert to polar form).

17. $(5 + 4i) + (-3 + 2i)$
18. $(-11 + 3i) - (-1 + 4i)$
19. $(3 + 2i) + (-6i) - (2 + 3i) - 10$
20. $(7 + 4i) - (-13 + 4i)$
21. $13 - 7i + 3i - 4$
22. $-8 - (4 + 3i) + 2i - (-6 - i)$
23. $(15 + 4i)(3 + 12i)$
24. $(-2 + 5i)(-3 - 5i)$
25. $-5i(6 - 4i)$
26. $(4 - 3i)^2$
27. $(2 - 3i)^3$
28. $i(2i)(3i)(-i)$
29. $\dfrac{5 + 4i}{5 + 2i}$
30. $\dfrac{-2 + i}{2 + 4i}$
31. $\dfrac{3 + 6i}{-2 - i}$
32. $\dfrac{5 + i}{5i}$

Write the polar form of each complex number. Round the results to the nearest tenth.

33. $5 - 2i$
34. $\sqrt{2} + 3i$
35. $-1 + 3i$
36. $\sqrt{3} - 2i$
37. $-3 + 4i$
38. $13 - 9i$

Write the polar form of each complex number. Leave the result in exact form.

39. $\sqrt{3} + i$
40. $1 - \sqrt{3}i$
41. $3 + 3i$
42. $-1 + \sqrt{3}i$
43. $-1 - i$
44. $\sqrt{5} - \sqrt{5}i$
45. $5i$
46. $-3i$

Write the rectangular form of the following numbers. Round the results to the nearest tenth.

47. 3 cis 15° **48.** 5 cis 20° **49.** 4.5 cis 35° **50.** 10 cis 40°
51. $\sqrt{2}$ cis 315° **52.** 200 cis 8° **53.** 13.6 cis(−25°) **54.** 12 cis(−6°)

Write the rectangular form of the following numbers. Leave the result in exact form.

55. $\sqrt{3}$ cis 30° **56.** 4 cis 210° **57.** 10 cis 300° **58.** 6 cis 135°
59. $\sqrt{10}$ cis 180° **60.** 2 cis 90° **61.** $\sqrt{8}$ cis 315° **62.** 5 cis 240°

Multiply or divide the following complex numbers. Leave the result in polar form.

63. (5 cis 30°)(3 cis 45°) **64.** (2 cis 18°)(4.5 cis 100°) **65.** (5.4 cis 300°)(2 cis 300°)

66. (0.5 cis 230°)(80 cis 200°) **67.** $\dfrac{20 \text{ cis } 100°}{5 \text{ cis } 20°}$ **68.** $\dfrac{100 \text{ cis } 45°}{200 \text{ cis } 15°}$

69. $\dfrac{40 \text{ cis } 80°}{18 \text{ cis } 160°}$ **70.** $\dfrac{90 \text{ cis } 300°}{50 \text{ cis } 100°}$

Use De Moivre's theorem to compute the power indicated. Leave the answer in the form in which the problem is stated (polar or rectangular).

71. $(8 \text{ cis } 100°)^3$ **72.** $(5 \text{ cis } 10°)^4$ **73.** $(3 \text{ cis } 200°)^3$ **74.** $(2 \text{ cis } 300°)^5$

75. $(0.5 - 1.2i)^8$ (round to nearest tenth) **76.** $(0.8 + 0.6i)^{10}$ (round to nearest tenth)

77. Find the 3 cube roots of 8 in exact form.
78. Find the 4 fourth roots of −1 in exact form.
79. Find the 4 fourth roots of 81 in exact form.
80. Find the 6 sixth roots of −64 in exact form.
81. Find the 3 cube roots of $75 - 100i$ to the nearest tenth.
82. Find the 4 fourth roots of $\sqrt{3} + 3i$ to the nearest tenth.

83. In electronics, one version of Ohm's law says that $I = \dfrac{V}{Z}$, where I is current, V is voltage, and Z is impedance. Find I in a circuit in which V is 125 cis 25° and Z is 50 cis 45°.

84. Find I in a circuit in which $V = 200$ cis 40° and $Z = 4$ cis 50°. See problem 83.

85. Find V in a circuit where $I = 10$ cis 15° and $Z = 5$ cis 30°. See problem 83.

86. Find Z in a circuit where $I = 40$ cis 200° and $V = 10$ cis 125°. See problem 83.

87. In a parallel electronics circuit with two legs, total impedance Z_T is $\dfrac{Z_1Z_2}{Z_1 + Z_2}$. Find Z_T in a circuit in which $Z_1 = 2 + i$ and $Z_2 = 3 - 5i$. Leave the answer in polar form.

88. Find Z_T in a parallel circuit in which $Z_1 = 12 + 3i$ and $Z_2 = 4 - 2i$. Leave the answer in polar form. See problem 87.

89. Use $I = \sqrt{\dfrac{P}{Z}}$ to determine I if $P = 5 + 2i$ and $Z = 1 - 4i$. Leave the result in rectangular form, to the nearest hundredth. Use the first square root ($k = 0$ in De Moivre's theorem).

90. Use $I = \sqrt{\dfrac{P}{Z}}$ to determine I if $P = -2 + 2i$ and $Z = 2 - i$. Leave the result in rectangular form, to the nearest hundredth. Use the first square root ($k = 0$ in De Moivre's theorem).

91. Is the following an identity: $a \text{ cis}(-\theta) = -a \text{ cis } \theta$? Show why or why not.

Multiplication by i can be interpreted as a 90° rotation. If z represents the complex number given in each of the following problems, compute and graph (a) z, (b) iz, (c) i^2z, (d) i^3z.

92. $4 + 2i$ **93.** $-3 + i$ **94.** $5i$ **95.** 6 **96.** $1 - i$ **97.** $-1 - i$

98. Let $z_1 = -2 + 2i$, $z_2 = 1 - \sqrt{3}i$. Form the product in two ways: (a) by multiplication in rectangular form and (b) by changing each value into polar form (in exact form) and performing the multiplication in polar form. Then (c) convert the answer to (b) back to rectangular form and verify that the answers to (a) and (b) are the same.

99. A numerically controlled machine is programmed to rotate a laser beam according to mathematical rules. The laser initially points to the point $1 + i$.

a. Find the rectangular form of a complex number z such that the angle of the product $z(1 + i)$ is 30° greater than the angle of $1 + i$, without changing the modulus of $1 + i$.

b. Give the rectangular form of the point to which the laser points after this rotation. Round the answer to two decimal places.

c. Give the rectangular form of the point to which the laser points after eight such rotations, starting at the point $1 + i$. Round the answer to two decimal places.

$\boxed{\textbf{100.}}$ In this section we stated that $\dfrac{r_1 \text{ cis } \theta_1}{r_2 \text{ cis } \theta_2} = \dfrac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$. Prove that this is true. Use the proof in the text that $(r_1 \text{ cis } \theta_1)(r_2 \text{ cis } \theta_2) = r_1 r_2 \text{ cis}(\theta_1 + \theta_2)$ as a guide.

101. *Warning:* This problem requires a *great* deal of algebraic manipulation. De Moivre's theorem for roots states that the n nth roots of r cis θ are of the form

$$(r \text{ cis } \theta)^{\frac{1}{n}} = r^{\frac{1}{n}} \text{ cis}\left(\frac{\theta}{n} + \frac{k \cdot 360°}{n}\right),\ 0 \le k < n,$$

where k and n are positive integers. Show that if $k \ge n$, then the expression is a repetition of another root. That is, it is the same as the expression for some value of $k < n$. Do this in the following manner.

First, if $k \ge n$, then $\dfrac{k}{n} = a + \dfrac{b}{n}$, where $b < n$. (Think of the example $22 \ge 5$, so $\frac{22}{5} = 4 + \frac{2}{5}$.) This means $k = an + b$, $b < n$.

Next, show that the following steps are true:

$$r^{\frac{1}{n}} \text{ cis}\left(\frac{\theta}{n} + \frac{k \cdot 360°}{n}\right)$$

$$= r^{\frac{1}{n}} \text{ cis}\left[\frac{\theta}{n} + \frac{(an + b) \cdot 360°}{n}\right] \qquad \text{Why?}$$

$$= r^{\frac{1}{n}} \text{ cis}\left[\left(\frac{\theta}{n} + \frac{b \cdot 360°}{n}\right) + a \cdot 360°\right].$$

Show the algebra for this step

Finally, now expand this last expression into terms of sine and cosine (i.e., expand "cis α"), and then apply the identities for the cosine and sine of a sum (section 5–2.).

7–2 Polar coordinates

Some natural phenomena, such as the motion of the planets, the contour of a cam on an automobile camshaft, the path traveled by a client on many of the rides in an amusement park, or the field strength around a radio transmitter, can be described most simply by describing the motion in terms of distance from some point as a line moves in a circle. The polar coordinate system is a coordinate system that is better suited to describing these phenomena than are rectangular coordinates. James Bernoulli is often credited with the creation of polar coordinates in 1691, although Isaac Newton used them earlier. A polar coordinate system is a series of concentric circles and an angle reference line (see figure 7–3). The common center of the circles is called the **pole.**

Basic definitions

Polar coordinates

The polar coordinates of a point is an ordered pair of the form (r, θ), where r is the *radius,* and θ is an angle, often measured in radians.

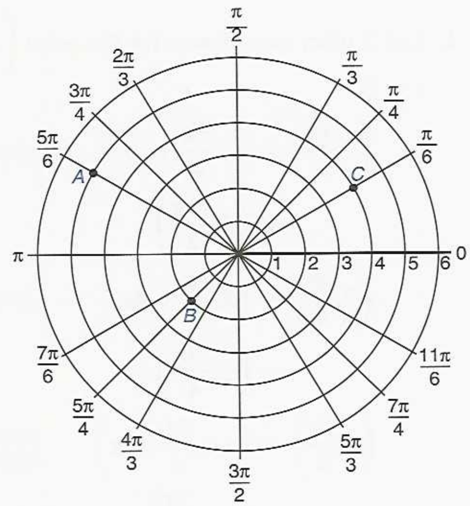


Figure 7–3

A point in polar form is located by finding the radius line corresponding to the angle θ, often stated in radians, and moving r units from the center along this line. Figure 7–3 shows the graphs[2] of the points $\left(5,\frac{5\pi}{6}\right)$ (A), $\left(2,\frac{5\pi}{4}\right)$ (B), and $\left(4,\frac{\pi}{6}\right)$ (C). In each of these cases $r > 0$.

If two points have equal radii and coterminal angles they will have the same graph. For this reason we define such points to be equivalent. If $r < 0$ we interpret this to mean a change of direction by π radians (the opposite direction).

Equivalence of points

1. $(r,\alpha) = (r,\beta)$ if α and β are coterminal angles.
2. $(-r,\theta) = (r,\theta + \pi)$.

Note $(-r,\theta) = (r,\theta - \pi)$ is also true.

This definition means that $(1,0)$, $(1,2\pi)$, $(1,4\pi)$, $(1,-2\pi)$, $(1,-4\pi)$, $(-1,\pi)$, $(-1,3\pi)$, $(-1,-\pi)$, etc. all describe the same point! This can occasionally lead to some confusion.

Example 7–2 A illustrates the basic definitions.

[2]Polar coordinate paper is widely available.

Example 7–2 A

1. List 3 other coordinates for the point $\left(2,\frac{3\pi}{4}\right)$, with at least one so that $r < 0$.

$$\left(2,\frac{3\pi}{4}\right) = \left(2,\frac{3\pi}{4} + 2\pi\right) = \left(2,\frac{11\pi}{4}\right)$$

Adding 2π gives a coterminal angle

$$\left(2,\frac{3\pi}{4}\right) = \left(2,\frac{3\pi}{4} + 4\pi\right) = \left(2,\frac{19\pi}{4}\right)$$

Adding 4π gives a coterminal angle

$$\left(2,\frac{3\pi}{4}\right) = \left(-2,\frac{3\pi}{4} + \pi\right) = \left(-2,\frac{7\pi}{4}\right)$$

Adding or subtracting π gives a coterminal angle in which r changes sign

2. Plot the point $\left(-5,\frac{11\pi}{6}\right)$.

$\left(-5,\frac{11\pi}{6}\right) = \left(5,\frac{11\pi}{6} - \pi\right) = \left(5,\frac{5\pi}{6}\right)$, which is plotted at A in figure 7–3.

Polar-rectangular coordinate conversions

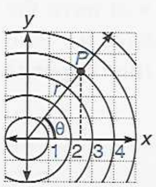


Figure 7–4

There is a way to relate polar and rectangular coordinates. Figure 7–4 shows polar and rectangular coordinates superimposed. From the definitions of section 2–2 we know that if $P = (x,y) = (r,\theta)$, $r > 0$, then $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$.

Thus, *to convert from polar to rectangular coordinates* we have only to use

$$x = r\cos\theta$$
$$y = r\sin\theta$$

In fact, it will be an exercise to show that we can use the same relations when $r < 0$. Example 7–2 B illustrates a polar to rectangular conversion.

Example 7–2 B

Convert the polar coordinates $\left(2,\frac{\pi}{3}\right)$ to rectangular coordinates.

$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2\cdot\frac{1}{2} = 1$$
$$y = r\sin\theta = 2\sin\frac{\pi}{3} = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$$

Thus, the rectangular coordinates are $(1,\sqrt{3})$.

To convert from rectangular to polar coordinates we use the fact that $r^2 = x^2 + y^2$ and $\tan\theta = \dfrac{y}{x}$ if $x \neq 0$. This also means $\theta' = \tan^{-1}\dfrac{y}{x}$.

As with vectors and complex numbers (sections 6–3 and 7–1) we always leave the angle θ so that it is the smallest possible absolute value. In this case, using radian measure, we always choose θ so that $-\pi < \theta \leq \pi$.

A rule for finding θ is essentially the same as that for finding θ_V for vectors, and θ for complex numbers.

> Given polar coordinates for point P, $P = (x,y) = (r,\theta)$,
>
> $$r = \sqrt{x^2 + y^2}, \quad \tan\theta' = \tan^{-1}\frac{y}{x} \text{ if } x \neq 0 \text{, and}$$
>
> $$\theta = \begin{cases} \theta' & \text{if } x > 0 \\ \theta' - \pi \text{ if } \theta' > 0 & \text{if } x < 0 \\ \theta' + \pi \text{ if } \theta' < 0 & \end{cases}$$

Note If $x = 0$ then θ is π if $y > 0$, and $-\pi$ if $y < 0$. This is clear from a sketch.

Example 7–2 C

Convert the rectangular coordinates into polar coordinates.

1. $(-2\sqrt{3},2)$

$r^2 = (-2\sqrt{3})^2 + 2^2 = 16,\ r = 4$

$$\theta' = \tan^{-1}\frac{2}{-2\sqrt{3}} = \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = -\tan^{-1}\frac{\sqrt{3}}{3}$$

$\tan^{-1}$ is an odd function, so $\tan^{-1}(-x) = -\tan^{-1}x$

so $\theta' = -\dfrac{\pi}{6}$. $x < 0$, $\theta' < 0$, so $\theta = \theta' + \pi = -\dfrac{\pi}{6} + \pi = \dfrac{5\pi}{6}$.

Therefore, the required polar coordinates are $\left(4,\dfrac{5\pi}{6}\right)$.

2. $(-2,-5)$

$$r^2 = (-2)^2 + (-5)^2 = 29,\ r = \sqrt{29} \approx 5.39$$
$$\theta' = \tan^{-1}\tfrac{5}{2} \approx 1.19 \text{ (radians)}$$

$x < 0$, $\theta' > 0$, so $\theta = \theta' - \pi = \tan^{-1}\frac{5}{2} - \pi$ (exactly), or approximately $1.19 - \pi \approx -1.95$. Thus, the polar coordinates are $(\sqrt{29}, \tan^{-1}2.5 - \pi)$ exactly, or approximately $(5.39,-1.95)$.

Conversions with a calculator

Engineering/scientific calculators are programmed to perform the conversions of example 7–2 C, using the same method and keys shown in section 6–3 for vectors and in section 7–1 for complex numbers. These calculators have keys marked R $\rightarrow$ P and P $\rightarrow$ R, or something equivalent. The results are stored in

locations referred to as x and y. Typical keystrokes are illustrated here. For these examples we want the *calculator in radian mode.* The steps for the TI-81 are shown below.

Example 7–2 B would be done as follows:

$\left(2, \frac{\pi}{3}\right)$ (polar)

2 $\boxed{P \to R}$ $\boxed{(}$ $\boxed{\pi}$ $\boxed{\div}$ 3 $\boxed{)}$ $\boxed{=}$ Display: $\boxed{1}$

$\boxed{x \leftrightarrow y}$ Display: $\boxed{1.732050808}$

Thus, $\left(2, \frac{\pi}{3}\right) \approx (1,1.73)$.

Example 7–2 C, part 2, would be done in the following way:

$(-2,-5)$ (rectangular)

2 $\boxed{\pm}$ $\boxed{R \to P}$ 5 $\boxed{+/-}$ $\boxed{=}$ Display: $\boxed{5.385164807}$

$\boxed{x \leftrightarrow y}$ Display: $\boxed{-1.951302704}$

Thus, $(-2,-5)$ (rectangular) $\approx (5.39,-1.95)$ (polar).

The TI-81 uses the values X, Y, R, and θ. (Y, R, θ are $\boxed{\text{ALPHA}}$ 1, $\boxed{\times}$, and 3, respectively.) It also uses the two MATH functions ''R ▶ P('' (rectangular to polar) and ''P ▶ R('' (polar to rectangular).

Example 7–2 B:

$\boxed{\text{MODE}}$ Rad $\boxed{\text{ENTER}}$ Make sure the calculator is in radian mode.

$\boxed{\text{MATH}}$ 2 2 $\boxed{\text{ALPHA}}$ $\boxed{.}$ $\boxed{(}$ π $\boxed{\div}$ 3 $\boxed{)}$

$\boxed{)}$ $\boxed{\text{ENTER}}$ Display: $\boxed{1}$

$\boxed{\text{ALPHA}}$ 1 $\boxed{\text{ENTER}}$ Display: $\boxed{1.732050808}$

Example 7–2 C, part 2:

$\boxed{\text{MATH}}$ 1 $\boxed{(-)}$ 2 $\boxed{\text{ALPHA}}$ $\boxed{.}$ $\boxed{(-)}$ 5

$\boxed{)}$ $\boxed{\text{ENTER}}$ Display: $\boxed{5.385164807}$

$\boxed{\text{ALPHA}}$ 3 $\boxed{\text{ENTER}}$ Display: $\boxed{-1.951302704}$

Conversion of equations between rectangular and polar form

Analytic geometry is the modeling of geometry in the language of algebra. Thus, a nonvertical line in analytic geometry is an equation of the form $y = mx + b$, and a circle with center at the origin and radius r is an equation of the form $x^2 + y^2 = r^2$.

Equations can also be written using polar coordinates. Examples are

$$r = 3 \cos \theta$$
$$r^2 = \sec \theta$$
$$r = 2$$
$$\cos \theta = 1$$

It is often useful to discover the polar coordinate version of a rectangular coordinate equation. We say that *a rectangular equation and a polar equation are equivalent if they describe the same set of points,* assuming the appropriate rectangular/polar conversions of the points themselves.

> **Conversion of equations from rectangular to polar form**
> To convert an equation in rectangular form into an equivalent equation in polar form, use the relations used to convert a point from rectangular to polar form:
> $$x = r\cos\theta,\ y = r\sin\theta, \text{ and } r^2 = x^2 + y^2.$$

When possible we customarily write a polar equation in which r is described as a function of θ. That is, to the extent possible, we put all terms with r in one member of the equation, and all other terms in the other member.

Example 7–2 D illustrates conversions from rectangular to polar form.

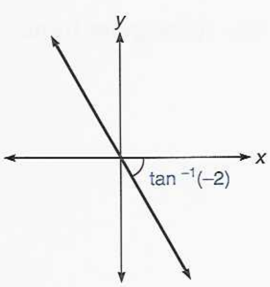
Example 7–2 D

Convert each rectangular equation into polar form.

1. The line $y = 3x - 2$.

$$y = 3x - 2$$
$$r\sin\theta = 3(r\cos\theta) - 2 \qquad x = r\cos\theta,\ y = r\sin\theta$$
$$2 = 3r\cos\theta - r\sin\theta \qquad \text{It is customary to solve a polar equation for } r \text{ if possible}$$
$$2 = r(3\cos\theta - \sin\theta)$$
$$r = \frac{2}{3\cos\theta - \sin\theta}$$

2. The line $y = -2x$.



$$y = -2x$$
$$r\sin\theta = -2r\cos\theta \qquad x = r\cos\theta,\ y = r\sin\theta$$
$$\sin\theta = -2\cos\theta \qquad \text{Divide both members by } r\text{, this assumes } r \neq 0 \text{ (see below)}$$
$$\frac{\sin\theta}{\cos\theta} = -2 \qquad \text{Divide both members by } \cos\theta\text{; this assumes } \cos\theta \neq 0 \text{ (see below)}$$
$$\tan\theta = -2$$

This equation is equivalent to the equation $y = -2x$. To see this, first observe that it does not mention r. This means r can be any value. The values of θ for which $\tan\theta = -2$ are in quadrants II and IV, where the tangent function takes on negative values. All points (r,θ) for which $\tan\theta = -2$ and r takes on any value are shown in the figure. This is the line $y = -2x$.

It was all right to assume $r \neq 0$ above because the resulting solution includes the pole as a solution (this is where $r = 0$).

It was also valid to assume $\cos\theta \neq 0$, for two reasons. One is that we arrive at an equation that satisfies the requirements and thus we do not need to consider the case where $\cos\theta = 0$. Second, when $\cos\theta = 0$, $\sin\theta$ is ± 1. These values do not solve the equation $\sin\theta = -2\cos\theta$, so that $\cos\theta = 0$ cannot occur in this situation.

3. The circle $x^2 + y^2 = 1$.

$$x^2 + y^2 = 1$$
$$r^2 = 1 \qquad x^2 + y^2 = r^2$$
$$r = \pm 1$$

Either $r = 1$ or $r = -1$ describes a circle with radius one, since θ is not restricted but may take on any value. Thus, either equation is valid.

4. The hyperbola $x^2 - y^2 = 2$. (See footnote[3].)

$$x^2 - y^2 = 2$$
$$(r\cos\theta)^2 - (r\sin\theta)^2 = 2$$
$$r^2\cos^2\theta - r^2\sin^2\theta = 2$$
$$r^2(\cos^2\theta - \sin^2\theta) = 2$$
$$r^2\cos 2\theta = 2 \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$r^2 = \frac{2}{\cos 2\theta}$$
$$r^2 = 2\sec 2\theta$$

Thus, either $r^2\cos 2\theta = 2$ or $r^2 = 2\sec 2\theta$ are valid equations of the given rectangular equation.

Conversion of equations from polar to rectangular form

To convert from polar to rectangular coordinates we use the relations

$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad r^2 = x^2 + y^2$$

Example 7–2 E illustrates converting polar equations into rectangular form.

Example 7–2 E

Convert the polar equation into rectangular form.

1. $r = 5\sec\theta$

$$r = \frac{5}{\cos\theta}$$
$$r\cos\theta = 5$$
$$r\cdot\frac{x}{r} = 5 \qquad \cos\theta = \frac{x}{r}$$
$$x = 5$$

[3]Hyperbolas are not covered in this text. Any equation of the form $ax^2 - by^2 = c$, $a, b, c \neq 0$, and a and b have the same sign, is a hyperbola.

2. $r^2 = \cos 2\theta$

We do not have any relation for $\cos 2\theta$, so we use an identity to replace it.

$$r^2 = \cos^2\theta - \sin^2\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$r^2 = \left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2 \qquad \cos\theta = \frac{x}{r},\ \sin\theta = \frac{y}{r}$$

$$r^2 = \frac{x^2}{r^2} - \frac{y^2}{r^2}$$

$$r^4 = x^2 - y^2$$

$$(x^2 + y^2)^2 = x^2 - y^2 \qquad r^4 = (r^2)^2 = (x^2 + y^2)^2$$

3. $r = 1 + \cos\theta$

$$r = 1 + \frac{x}{r}$$

$$r^2 = r + x \qquad \text{Multiply each member by } r$$

$$x^2 + y^2 = r + x$$

$$x^2 + y^2 - x = r$$

$$(x^2 + y^2 - x)^2 = r^2 \qquad \text{Square both members to obtain } r^2$$

$$(x^2 + y^2 - x)^2 = x^2 + y^2 \qquad \text{Now replace } r^2 \text{ by } x^2 + y^2$$

$$x^4 - 2x^3 + y^4 + 2x^2y^2 - 2xy^2 - y^2 = 0 \qquad \text{Expand the left member and combine terms}$$

Note Observe that r can be replaced by squaring both members as necessary to obtain r^2, or any even power of r.

In example 7–2 E we used the identities $\sin\theta = \frac{y}{r}$ and $\cos\theta = \frac{x}{r}$. This is only valid if $r \neq 0$. The value of r is zero only if the pole is a solution to the given polar equation. Thus, when the pole is a solution to the polar equation we must verify that the point (0,0) is a solution to the resulting rectangular equation.

In example 7–2 E, part 1, the pole is not a solution, since if $r = 0$, $0 = 5 \sec\theta = \frac{5}{\cos\theta}$. The equation $0 = \frac{5}{\cos\theta}$ has no solution. (The rectangular equation $x = 5$ does not pass through the origin either.)

In example 7–2 E, part 2, r can take on the value 0: $0 = \cos 2\theta$ has solutions. Thus, the pole is part of this equation. Observe that the point (0,0) is also a solution to the equation $(x^2 + y^2)^2 = x^2 - y^2$.

Mastery points

Can you

- Graph points in the polar coordinate system?
- Give alternate polar coordinates for a given point?
- Convert between polar and rectangular coordinates?
- Convert equations between polar and rectangular form?

Exercise 7–2

Graph the following points in polar coordinates.

1. $(3,0)$ **2.** $(4,\pi)$ **3.** $\left(2,\frac{\pi}{6}\right)$ **4.** $\left(1,\frac{\pi}{3}\right)$ **5.** $\left(2,\frac{3\pi}{4}\right)$ **6.** $\left(3,\frac{7\pi}{6}\right)$

7. $\left(6,\frac{11\pi}{6}\right)$ **8.** $\left(5,\frac{\pi}{2}\right)$ **9.** $(-2,\pi)$ **10.** $\left(-4,\frac{5\pi}{3}\right)$ **11.** $\left(-1,\frac{\pi}{3}\right)$ **12.** $\left(-5,\frac{\pi}{6}\right)$

13. $(4,2)$ **14.** $(3,5)$ **15.** $(-4,6)$ **16.** $\left(-4,\frac{\pi}{2}\right)$ **17.** $\left(5,\frac{4\pi}{3}\right)$ **18.** $\left(4,\frac{5\pi}{6}\right)$

List three other coordinates for the point, with two points having $r > 0$ and one point having $r < 0$.

19. $\left(2,\frac{\pi}{6}\right)$ **20.** $\left(1,\frac{\pi}{3}\right)$ **21.** $\left(6,\frac{11\pi}{6}\right)$ **22.** $\left(-5,\frac{\pi}{6}\right)$ **23.** $(2,2)$ **24.** $\left(4,\frac{17\pi}{6}\right)$

Convert the following polar coordinates into rectangular coordinates. Leave the result in exact form.

25. $\left(4,\frac{\pi}{2}\right)$ **26.** $\left(2,\frac{\pi}{3}\right)$ **27.** $\left(5,\frac{5\pi}{6}\right)$ **28.** $\left(1,\frac{11\pi}{6}\right)$ **29.** $\left(4,\frac{4\pi}{3}\right)$ **30.** $\left(2,\frac{5\pi}{3}\right)$

Convert the following polar coordinates into rectangular coordinates to two-decimal place accuracy.

31. $(2,1)$ **32.** $(5,1.2)$ **33.** $(3,0.82)$ **34.** $(3,5)$ **35.** $(4,4)$ **36.** $(1,6)$

Convert the following rectangular coordinates into polar coordinates. Leave the result in exact form.

37. $(-2\sqrt{3},-2)$ **38.** $(3,-3)$ **39.** $(-2,0)$ **40.** $(-1,\sqrt{3})$ **41.** $(-4,-4)$ **42.** $(0,1)$

Convert the following rectangular coordinates into polar coordinates to two-decimal place accuracy.

43. $(2,3)$ **44.** $(-5,2)$ **45.** $(1,-4)$ **46.** $(-4,-3)$ **47.** $(5,4)$ **48.** $(-3,5)$

Convert the following rectangular equations into polar equations.

49. $y = 4x$ **50.** $y = -2x$ **51.** $y = -3x + 2$ **52.** $y = 5x - 3$

53. $y = mx + b, b \neq 0$ **54.** $y = 2$ **55.** $y^2 - 2x^2 = 5$ **56.** $y^2 - x = 4$

57. $3x^2 + 2y^2 = 1$ **58.** $x^2 + y^2 = 3$

Convert the following polar equations into rectangular equations.

59. $r = \sin\theta$ **60.** $2r = \cos\theta$ **61.** $r = 2\sec\theta$ **62.** $r = 3\csc\theta$

63. $r = 3\sin 2\theta$ **64.** $r = 2\cos\theta$ **65.** $r^2 = \sin 2\theta$ **66.** $r = \cos 2\theta$

67. $r^2 = \tan\theta$ **68.** $r\sin\theta = 5$ **69.** $r = \dfrac{3}{1 - 2\sin\theta}$ **70.** $r = \dfrac{5}{4 - \cos\theta}$

71. Show that a polar equation of $2xy = 5$ is $r^2 = 5 \csc 2\theta$.

72. In the text we noted that $x = r \cos \theta$ if $r > 0$. Show that this is also true for a point given in polar coordinates where $r < 0$. (*Hint:* Consider a point $P = (r,\theta)$, where $r < 0$. Then $P = (-r,\theta + \pi)$, where $-r > 0$. Therefore, since $-r > 0$, $x = -r \cos(\theta + \pi)$ is true. Proceed from here.)

73. In the text we noted that $y = r \sin \theta$ if $r > 0$. Show that this is also true for a point given in polar coordinates where $r < 0$. See the hint in problem 72.

74. The shape of a cam that drives a certain sewing machine needle is described by the polar equation $r = 3 - 2 \cos \theta$. Convert this equation into rectangular form.

75. The path that an industrial robot must follow to paint a pattern on a part being manufactured is described by the curve $r = 1 - 2 \sin \theta$. Convert this equation into rectangular form.

76. The pattern of strongest radiation of a certain bidirectional radio antenna is described by the curve $r = 1 + \sin 2\theta$. Convert this equation into rectangular form.

77. The pattern of strongest radiation of a certain radio antenna is described by the equation $r = 1 + 2 \sin 2\theta$. This pattern is said to have side lobes. Convert this equation into rectangular form.

In the October 1983 issue of *Scientific American,* Jearl Walker described several rides, the Scrambler and the Calypso, at the Geauga Lake Amusement Park near Cleveland, Ohio.

$\boxed{\textbf{78.}}$ Assume the path taken by the Scrambler is described by the polar equation $r = 2 \cos 3\theta$. Convert this equation into rectangular form. It will be necessary to rewrite $\cos 3\theta$ in terms of $\cos \theta$. See problem 79 in section 5–3.

79. Assume the path of the Calypso is described by the equation $r = 1 - 3 \cos \theta$. Convert this equation into rectangular form.

7–3 Graphs of polar equations

There are several ways to graph polar equations. The graphing calculator can be used, or one can sketch a graph by hand. We illustrate how to obtain polar graphs with the TI-81 graphing calculator. To achieve a sketch of a graph by hand we employ three methods:

1. plotting points,
2. putting a polar equation into rectangular form, and
3. using the corresponding rectangular graph as a guide.

The examples show how to create polar graphs both by hand and by graphing calculator.

Graphing polar equations with the graphing calculator

The TI-81 uses a mode called Parametric Mode to graph polar equations. Parametric mode uses two functions to produce rectangular coordinates. The first produces the x-coordinates, and the second produces the y-coordinates. We illustrate how to use the process by graphing the polar equation $r = \sin^2\theta$.

The polar equation $r = \sin^2\theta$ generates a collection of polar coordinates (r,θ). In this case, r is $\sin^2\theta$, so the polar coordinates generated are $(\sin^2\theta,\theta)$. To convert a polar coordinate to a rectangular coordinate we map (r,θ) to (x,y) $= (r \cos \theta, r \sin \theta)$ (section 7–2), so for this function we map $(\sin^2\theta,\theta)$ to the point $(x,y) = (\sin^2\theta \cdot \cos \theta, \sin^2\theta \cdot \sin \theta)$.

As stated above, in parametric mode a point (x,y) is calculated from two separate equations that depend on a third variable, usually called t (the TI-81 uses T). Thus, we graph the set of rectangular coordinates $(\sin^2 t \cdot \cos t, \sin^2 t \cdot \sin t)$, which of course is the same as $(\sin^2 t \cdot \cos t, \sin^3 t)$. We usually let t take on the values from 0 to 2π to obtain the graph, but this may need to be adjusted depending on the functions in question.

As illustrated above, to graph a polar function of the form $r = f(\theta)$, put the calculator in parametric mode and graph

$$X_{1T}=f(T) \cdot \cos T$$
$$Y_{1T}=f(T) \cdot \sin T$$

To create the graph of the polar equation $r = \sin^2\theta$ do the following:

[MODE] Select Param instead of Function. Do this by positioning the blinking square over Param and using [ENTER]. Also select Polar instead of Rect. This has no effect except when using the trace feature. Then the values T, R and θ are shown instead of T, X and Y.

[Y=] The display now shows the variables X_{1T} and Y_{1T} (as well as others). Enter the equations shown above. Note that in parametric mode the [X|T] key is used to enter T. The display should look like :X_{1T}=(sin T)^2*cos T
:Y_{1T}=(sin T)^3.
Remember, the exponent 3 is [MATH] 3.

[RANGE] The range display is different in parametric mode. Set the following values: Tmin=0
Tmax=6.28 (2π)
Tstep=.105 $\left(\frac{\pi}{30}\right)$
Xmin=−1
Xmax=1
Xscl=.5
Ymin=−1
Ymax=1
Yscl=.5

[ZOOM] 5 We want the display to be square.

The graph is drawn as shown in figure 7–5.

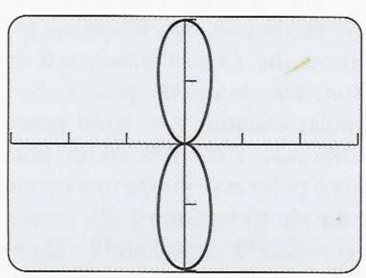


Figure 7–5

We show the range settings in the rest of this section in the order shown above. Thus, for the graph above we show

X_{1T}=(sin T)^2*cos T, Y_{1T}=(sin T)^3,
$\boxed{\text{RANGE}}$ 0,6.28,0.105,−1,1,.5,−1,1,.5, $\boxed{\text{ZOOM}}$ 5

Graphing polar equations by plotting points

Of course any graph can be obtained by plotting enough points. Example 7–3 A shows a case where this might be done without plotting too many points.

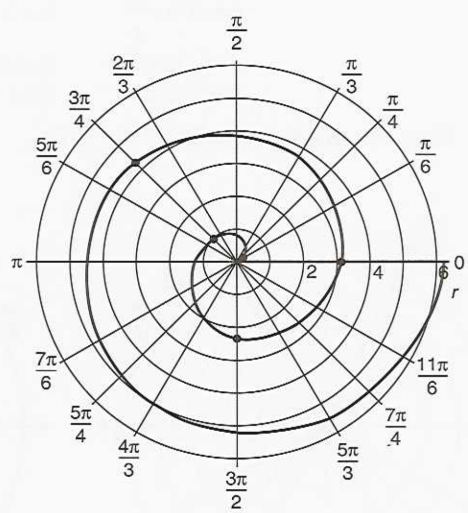
■ **Example 7–3 A**

Graph the polar equation $r = \frac{\theta}{2}$, $\theta \geq 0$.

We observe that r increases at half the rate of θ. A table of values is given in table 7–1, and these values are plotted in the figure.

θ	r
$\frac{\pi}{4}$	$\frac{\pi}{8} \approx 0.4$
$\frac{3\pi}{4}$	$\frac{3\pi}{8} \approx 1.2$
$\frac{3\pi}{2}$	$\frac{3\pi}{4} \approx 2.4$
2π	$\pi \approx 3.1$
$\frac{11\pi}{4}$	$\frac{11\pi}{8} \approx 4.3$

Table 7–1



The graph shows points (r,θ) for $0 \leq \theta \leq 4\pi$, but the graph, a spiral, continues on forever.

X_{1T}=T/2*cos T, Y_{1T}=T/2*sin T,
$\boxed{\text{RANGE}}$ 0,20,0.105,−6,6,1,−6,6,1, $\boxed{\text{ZOOM}}$ 5

The next two methods presented can save the time required to calculate the many points that are often required to graph an equation by plotting points.

Graphing polar equations by putting in rectangular form

■ *Example 7–3 B*

Graph the polar equation $\theta = 2$.

An angle of 2 (radians) is shown in part (a) of the figure. Any point $(r,2)$, $r > 0$, lies on the terminal side of this angle.

Recall that $(-r,\theta) = (r,\theta + \pi)$. Therefore $(-r,2) = (r,2 + \pi)$. Thus, when r is negative, its graph is along the line containing the terminal side of the angle $\theta = 2 + \pi$. Thus, the graph of $\theta = 2$ is the line shown in part (b) of the figure.

Another way to see the shape of this graph is to *rewrite the equation in rectangular form.*

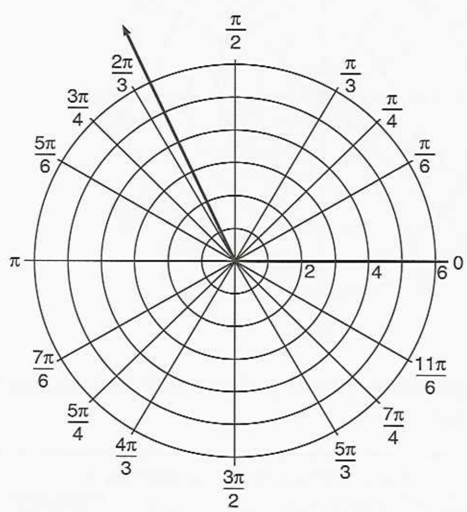
$$\tan \theta = \frac{y}{x}$$

$$\tan 2 = \frac{y}{x} \qquad \theta = 2$$

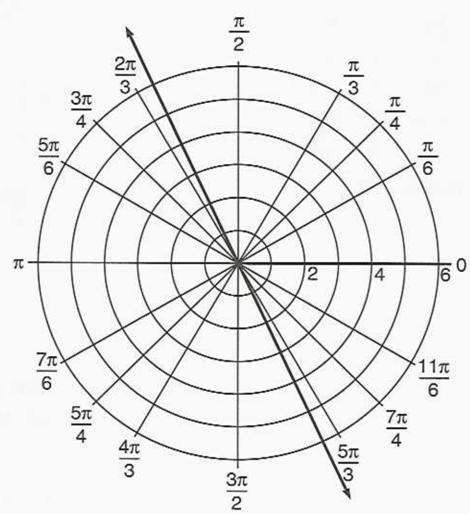
$$y = (\tan 2)x \qquad \text{Straight line, slope} = \tan 2$$

$$y \approx -2.2x \qquad \tan 2 \approx -2.2$$

Thus, the graph is the line $y \approx -2.2x$.



(a)



(b)

The equation $\theta = 2$ does not describe r as a function of θ, since r is not even mentioned! This means that r can take on any value. Thus, any point in the graph is of the form $(r,2)$. Parametrically we let $r = T$ and $\theta = 2$.

X_{1T}=T*cos 2, Y_{1T}=T*sin 2,

$\boxed{\text{RANGE}}$ −7,7,0.105,−6,6,1,−6,6,1, $\boxed{\text{ZOOM}}$ 5

Graphing polar equations using the rectangular form as a guide

The rectangular form of an equation can provide guidance for drawing the polar form of an equation. This is especially true when the rectangular form is periodic. There are two principles involved in this process. To discuss this we define the term ''positive lobe.'' A *positive lobe* is a portion of a graph in *rectangular* coordinates that starts and ends on the x-axis, and is continuous (it has no breaks, such as a vertical asymptote). Figure 7–6 (a) shows two positive lobes, at A and at B.

Every positive lobe corresponds to a lobe in a polar graph. A lobe (in a polar graph) is a closed figure that starts and ends at the pole. Figure 7–6 shows the correspondence between two positive lobes A and B in a rectangular graph (a), and the corresponding lobes in a polar graph (b).

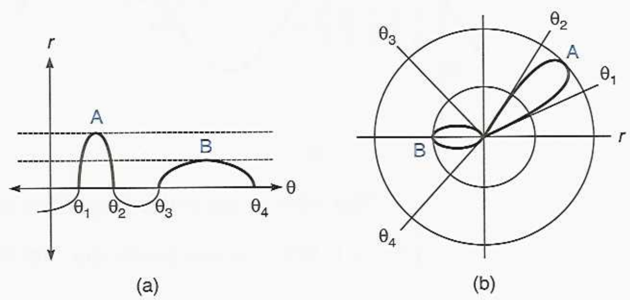


Figure 7–6

Another important point uses the fact that $(-r,\theta)$ has the same graph as $(r,\theta \pm \pi)$. This means that negative lobes in a rectangular graph may be redrawn as positive lobes by shifting their graphs $\pm\pi$ units, before graphing in polar coordinates. Although this method sometimes seems complicated, with a little practice it is much easier than having to plot many points. Example 7–3 C illustrates this method.

■ ***Example 7–3 C***

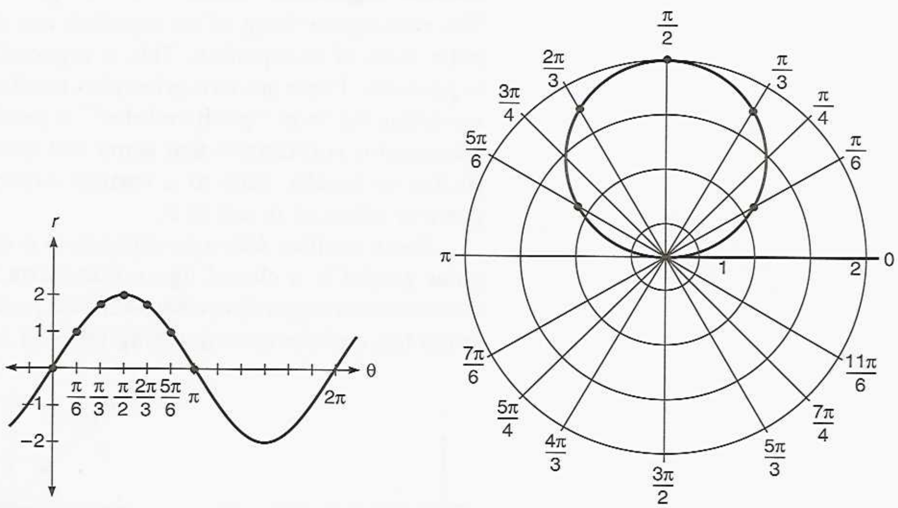
Graph the polar equation, using the rectangular form as a guide.

1. $r = 2 \sin \theta$

Part (a) of the figure is the graph of $r = 2 \sin \theta$, $0 \leq \theta \leq 2\pi$, in rectangular coordinates (as done in chapter 3). We observe that there is a positive lobe from $0 \leq \theta \leq \pi$. This lobe is graphed in part (b) of the figure. Table 7–2 shows values plotted in both graphs. The negative lobe from π to 2π gives the same graph in polar form as the positive lobe. This is because if we shifted the negative lobe π units to the left and made it positive, it would be the same as the positive lobe.

θ	$2 \sin \theta$
0	0
$\frac{\pi}{6}$	1
$\frac{\pi}{3}$	1.7
$\frac{\pi}{2}$	2
$\frac{2\pi}{3}$	1.7
$\frac{5\pi}{6}$	1
π	0

Table 7–2



(a) (b)

The lobe in the polar graph is a circle with center at the polar point $\left(\frac{\pi}{2}, 1\right)$. We will not prove the fact that the graph is actually a circle.

X_{1T}=2*sin T*cos T, Y_{1T}=2*sin T*sin T,
$\boxed{\text{RANGE}}$ 0,3.14,0.105,−2,2,1,−2,2,1, $\boxed{\text{ZOOM}}$ 5

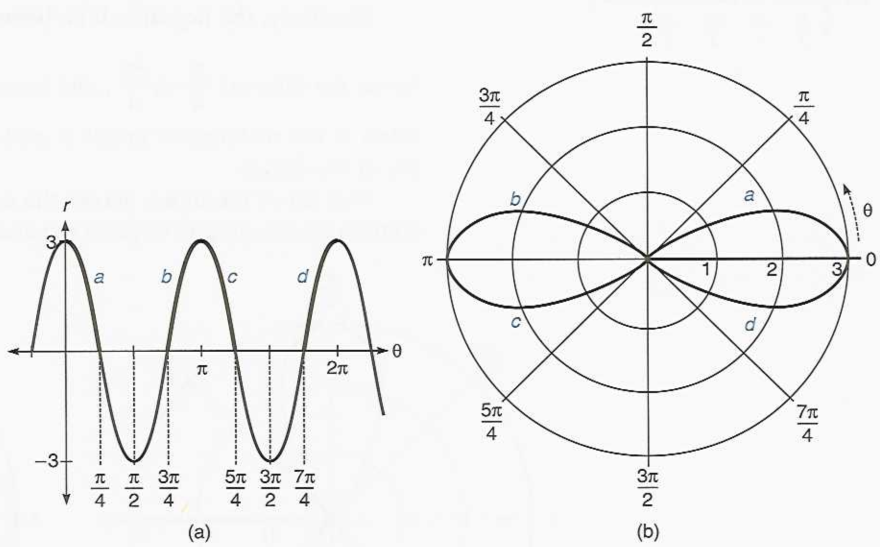
2. $r = 3 \cos 2\theta$

First graph in rectangular coordinates (as in section 3–3).

$$0 \le 2\theta \le 2\pi$$

$$0 \le \theta \le \pi \qquad \text{Divide each member by 2}$$

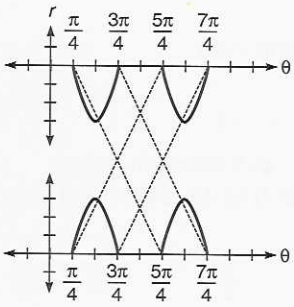
We get one basic cosine cycle, with amplitude 3, as θ takes on values from 0 to π. There are thus two basic cycles from 0 to 2π. Part (a) of the figure shows this graph.



Part (b) of the figure is obtained from part (a) in the following manner. Part (a) shows that as θ goes from 0 to $\frac{\pi}{4}$, r goes from 3 down to 0. In part (b), $r = 3$ when $\theta = 0$. As θ increases to $\frac{\pi}{4}$, r moves down to 0. This produces the half of a lobe labeled *a*.

Part (a) also shows that as θ goes from $\frac{3\pi}{4}$ to π, r goes from 0 to 3. This produces the half lobe *b* in part (b) of the figure. As θ continues from π to $\frac{5\pi}{4}$ (part (a)), r goes from 3 back down to 0. This produces the effect at *c* in part (b).

The half lobe at *d* (part (b)) is produced from what r is doing at *d* in part (a).

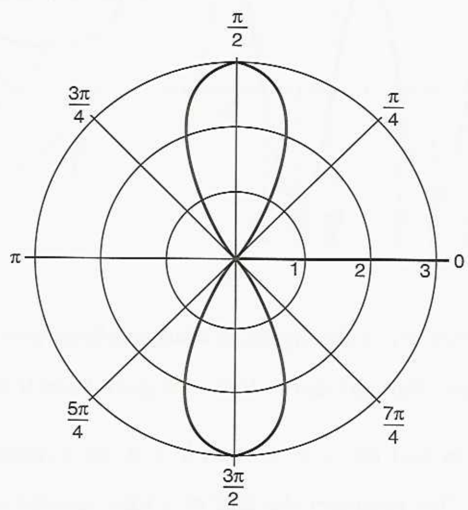


(c)

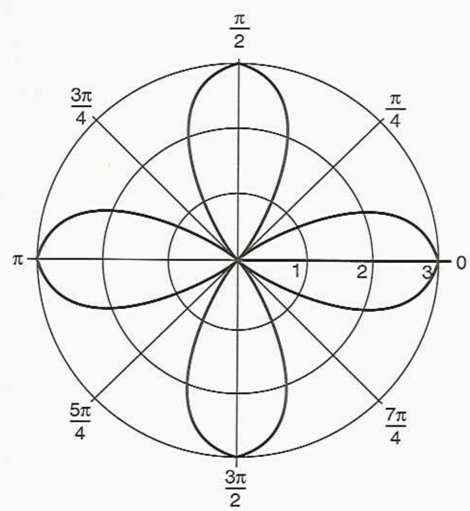
Part (c) of the figure shows one way to handle negative lobes in the rectangular graph. Shifting θ by π or $-\pi$ units causes r to change its sign; thus, we can shift the negative lobe between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$, by adding π, to the interval $\frac{5\pi}{4}$ to $\frac{7\pi}{4}$, and in the process the negative lobe becomes positive.

Similarly, the negative lobe between $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$ is shifted $-\pi$ units to lie on the interval $\frac{\pi}{4}$ to $\frac{3\pi}{4}$, and becomes a positive lobe. These positive lobes in the rectangular graph in part (c) produce the lobes shown in part (d) of the figure.

Part (e) of the figure shows the complete graph produced by combining the graphs of parts (b) and (d).

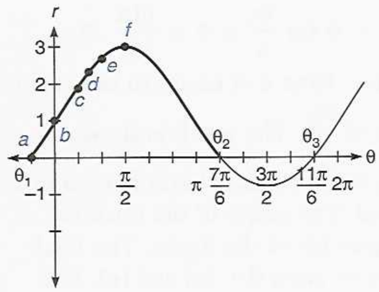


(d)



(e)

X_{1T}=3*cos2T*cos T, Y_{1T}=3*cos2T*sin T,
$\boxed{\text{RANGE}}$ 0,6.3,0.105,−3,3,1,−3,3,1, $\boxed{\text{ZOOM}}$ 5



(a)

3. $r = 1 + 2 \sin \theta$

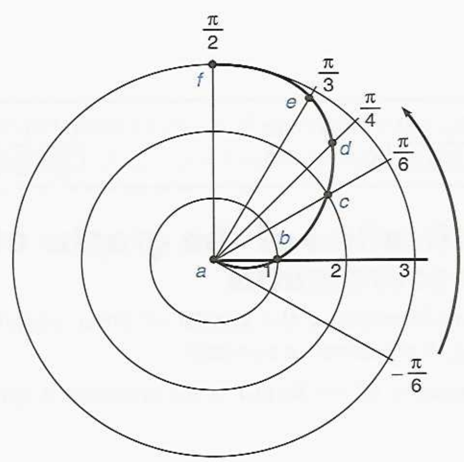
The graph of $r = 1 + 2 \sin \theta$ in rectangular coordinates is shown in part (a) of the figure. It is the graph of $y = 2 \sin \theta$, shifted up by one unit.

Note that r goes from 0 up to 3 and back to 0 between angles θ_1 and θ_2. It would help to find these two angles. They occur where $r = 0$, so we solve for θ where $r = 0$.

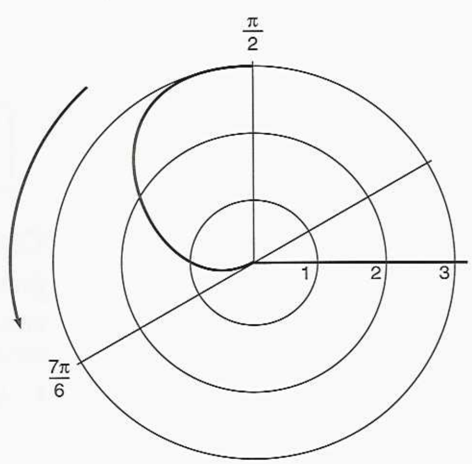
$$r = 1 + 2 \sin \theta$$
$$0 = 1 + 2 \sin \theta$$
$$-1 = 2 \sin \theta$$
$$-\tfrac{1}{2} = \sin \theta$$
$$\theta' = \sin^{-1}\tfrac{1}{2}$$
$$\theta' = \frac{\pi}{6}(30°)$$

θ is an angle in quadrant III or IV, since $\sin \theta < 0$. Thus, θ is $\pi + \frac{\pi}{6} = \frac{7\pi}{6}$ and $2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$. By subtracting the period, 2π, from $\frac{11\pi}{6}$ we can see that θ_1 is $-\frac{\pi}{6}$.

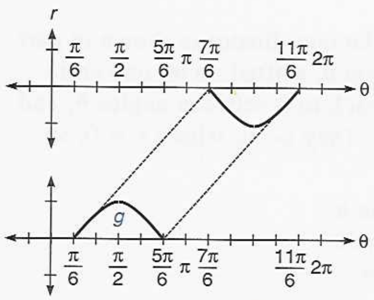
Now, referring to part (a) of the figure, we see that as θ goes from $-\frac{\pi}{6}$ to $\frac{\pi}{2}$, r goes from 0 to 3. This is shown in part (b) of the figure. Similarly, as θ goes from $\frac{\pi}{2}$ to $\frac{7\pi}{6}$, r goes from 3 back down to 0. This is shown in part (c).



(b)

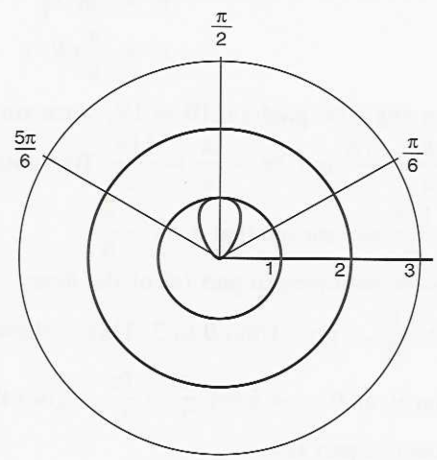


(c)

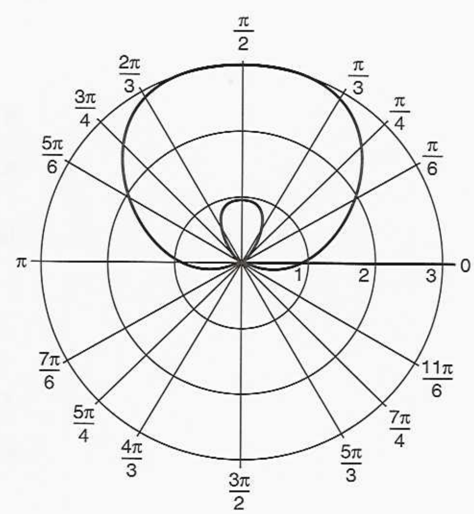


(d)

In part (a) of the figure we see that $r < 0$ for $\frac{7\pi}{6} \le \theta \le \frac{11\pi}{6}$. By adding or subtracting π, r becomes positive. Here it is easier to subtract π from each of these values, giving $\frac{\pi}{6} \le \theta \le \frac{5\pi}{6}$. The result is shown in rectangular coordinates in part (d), where we see that r varies from 0 to 1 and back to 0 as θ moves over this interval. The graph of the positive lobe g, in polar coordinates, is shown in part (e) of the figure. The final graph of $r = 1 + 2\sin\theta$ is a combination of parts (b), (c) and (e). It is shown in part (f) of the figure.



(e)



(f)

X_{1T}=(1+2sinT)*cos T, Y_{1T}=(1+2sinT)*sin T,
$\boxed{\text{RANGE}}$ 0,6.3,0.105,−3,3,1,−3,3,1, $\boxed{\text{ZOOM}}$ 5

Classification of the graphs of equations in polar coordinates

Some classification of the graphs of polar equations has been done. In the following, k represents a constant.

1. An equation of the form $r = k\theta$ produces a *spiral* (example 7–3 A).

2. An equation of the form $r \sin \theta = k$ or $r \cos \theta = k$ produces a *horizontal or vertical straight line* for its graph. When graphing these, it is easiest to put the equation in rectangular form. For example, $r \sin \theta = 2$ can be transformed, using $\sin \theta = \frac{y}{r}$.

$$r \sin \theta = 2$$
$$r \cdot \frac{y}{r} = 2$$
$$y = 2$$

This is a horizontal straight line.

3. Equations of the form $\theta = k$ also produce *straight lines* (example 7–3 B).
4. Equations of the form $r = k$ produce *circles with center at the pole.*
5. Equations of the form $r = k \cos \theta$ or $r = k \sin \theta$ also produce *circles, which pass through the pole, but have centers elsewhere* (example 7–3 C, part 1).
6. The graph of an equation of the form $r = k \cos n\theta$ or $r = k \sin n\theta$ is called a *rose.* It has n leaves if n is odd, and $2n$ leaves if n is even. Example 7–3 C, part 2, is a four-leafed rose.
7. Equations of the form $r = a + b \cos \theta$ or $r = a + b \sin \theta$ produce a figure called the *limaçon.* Example 7–3 C, part 3, is an example. If $|a| = |b|$, the graph is heart shaped and is called a *cardioid.*

Mastery points

Can you

- Graph a polar equation by plotting points?
- Graph a polar equation by using its rectangular graph as a guide?

Exercise 7–3

Graph the polar equation.

1. $r = 6$	2. $r = 2$	3. $r = -3$	4. $r = -1$
5. $\theta = 1$	6. $\theta = 3$	7. $\theta = \frac{\pi}{3}$	8. $\theta = -1$
9. $r \sin \theta = 6$	10. $r \cos \theta = 1$	11. $r \cos \theta - 2 = 0$	12. $r \sin \theta = -3$
13. $r = 3 \sin \theta$	14. $r = 2 \cos \theta$	15. $r = 4 \cos \theta$	16. $r = \sin \theta$
17. $r = 3 \sin 2\theta$	18. $r = 2 \cos 3\theta$	19. $r = 3 \cos 4\theta$	20. $r = \sin 3\theta$
21. $r = 1 + \sin \theta$	22. $r = 1 - \sin \theta$	23. $r = 1 - 2 \cos \theta$	24. $r = 2 - 3 \sin \theta$
25. $r = 2 - 2 \sin 2\theta$	26. $r = 1 - \cos 2\theta$	27. $r = 1 + \cos 3\theta$	28. $r = 2 + \sin 2\theta$
29. $r = \theta, \theta < 0$	30. $r = 1 + \theta, \theta > 0$	31. $r = \frac{\theta}{4}, \theta > 0$	32. $r = 1 + \frac{\theta}{2}, \theta > 0$

33. The shape of a cam that drives a certain sewing machine needle is described by the polar equation $r = 3 - 2\cos\theta$. Draw the cam by graphing this equation.

34. The path an industrial robot must follow to paint a pattern on a part being manufactured is described by the curve $r = 1 - 2\sin\theta$. Graph this equation.

35. The pattern of strongest radiation of a certain bidirectional radio antenna is described by the curve $r = 1 + \sin 2\theta$. Graph this pattern.

36. The pattern of strongest radiation of a certain radio antenna is described by the equation $r = 1 + 2\sin 2\theta$. This pattern is said to have side lobes. Graph the pattern.

As noted in section 7–2, in the October 1983 issue of *Scientific American,* Jearl Walker described several rides at the Geauga Lake Amusement Park near Cleveland, Ohio. The Scrambler is a ride whose motion could be described as a three-leafed rose, and the motion of the Calypso could be described as a limaçon.

37. Graph the path taken by the Scrambler, assuming that its motion is described by the polar equation $r = 2\cos 3\theta$.

38. Graph the path of the Calypso, assuming that its motion is described by the equation $r = 1 - 3\cos\theta$.

Chapter 7 summary

- **Imaginary unit** $i = \sqrt{-1}$
- **Complex number (rectangular form)** A number of the form $a + bi$, a and b both real numbers.
- **Complex conjugate** The complex conjugate of $a + bi$ is $a - bi$.
- **Equality of complex numbers** $a + bi = c + di$ if and only if $a = c$ and $b = d$.
- **Polar form of a complex number** If $z = a + bi$ is a complex number that determines an angle θ, then $r \text{ cis } \theta$ is its polar form, where $\text{cis } \theta$ means $\cos\theta + i\sin\theta$. The value r is called the modulus of z, which is also written $|z|$, and

$$r = \sqrt{a^2 + b^2}, \quad \tan\theta' = \tan^{-1}\frac{b}{a}, \text{ and}$$

$$\theta = \begin{cases} \theta' & \text{if } a > 0 \\ \theta' - 180° \text{ if } \theta' > 0 & \text{if } a < 0 \\ \theta' + 180° \text{ if } \theta' < 0 & \end{cases}$$

- **Complex multiplication—polar form** $(r_1 \text{ cis } \theta_1)(r_2 \text{ cis } \theta_2) = r_1r_2 \text{ cis}(\theta_1 + \theta_2)$.
- **Complex division—polar form** $\dfrac{r_1 \text{ cis } \theta_1}{r_2 \text{ cis } \theta_2} = \dfrac{r_1}{r_2} \text{ cis}(\theta_1 - \theta_2)$, $r_2 \neq 0$.
- **De Moivre's theorem** $(r \text{ cis } \theta)^n = r^n \text{ cis } n\theta$ for any real number n.
- **De Moivre's theorem for roots** The n nth roots of $r \text{ cis } \theta$ are of the form $(r \text{ cis } \theta)^{\frac{1}{n}} = r^{\frac{1}{n}} \text{ cis}\left(\dfrac{\theta}{n} + \dfrac{k \cdot 360°}{n}\right)$, $0 \le k < n$, where k and n are positive integers.
- The polar coordinates of a point is an ordered pair of the form (r,θ), where r is the radius and θ is an angle.
- **Equivalence of points in polar coordinates**
  1. $(r,\alpha) = (r,\beta)$ if α and β are coterminal angles.
  2. $(-r,\theta) = (r,\theta \pm \pi)$.
- **Relation between polar and rectangular coordinates** If $P = (x,y) = (r,\theta)$, $r > 0$, then $\cos\theta = \dfrac{x}{r}$ and $\sin\theta = \dfrac{y}{r}$. Thus, $x = r\cos\theta$ and $y = r\sin\theta$. Also,

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta' = \tan^{-1}\frac{y}{x}, \text{ and}$$

$$\theta = \begin{cases} \theta' & \text{if } x > 0 \\ \theta' - \pi \text{ if } \theta' > 0 & \text{if } x < 0 \\ \theta' + \pi \text{ if } \theta' < 0 & \end{cases}$$

- To convert a rectangular equation into a polar equation use the relations $x = r\cos\theta$, $y = r\sin\theta$, and $r^2 = x^2 + y^2$.
- To convert a polar equation into a rectangular equation use the relations $\sin\theta = \dfrac{y}{r}$, $\cos\theta = \dfrac{x}{r}$, and $r^2 = x^2 + y^2$.

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Chapter 7 review

[7–1] Identify the real and imaginary parts of the following complex numbers.

1. $3 - i$ **2.** $3i$ **3.** 12 **4.** $-i$

Simplify each expression by performing the indicated operations.

5. $(2 - 4i) + (-3 + 2i)$ **6.** $(-7 + 4i) - (-13 + 3i)$
7. $(5 - 4i)(3 + 12i)$ **8.** $(-2 + 5i)(3 - 6i)$
9. $4i(2 - 3i)$ **10.** $(4 - i)(2 - 3i)(5 + 2i)$
11. $\frac{10 - 4i}{5 + 2i}$ **12.** $\frac{-2 + i}{2 - 4i}$ **13.** $\frac{8 + 6i}{14i}$

If $z_1 = 2 + i$ and $z_2 = 3 - 2i$, evaluate the following expressions.

14. $2z_1 - z_2$ **15.** $z_1(3 - z_2)$
16. $\frac{z_1 + z_2}{z_1 - z_2}$ **17.** $\frac{z_1(z_1 - z_2)}{2z_2}$

In each of the following problems, (a) find the complex conjugate of the given number and (b) graph both the number and its complex conjugate in the same graph.

18. $-4 + 3i$ **19.** $-4 + i$ **20.** $3i$ **21.** 17

Write the polar forms of the following numbers (to the nearest tenth).

22. $3 - 2i$ **23.** $\sqrt{3} + 3i$ **24.** $-1 - 2i$

Write the polar forms of the following complex numbers. Leave the result in exact form.

25. $\sqrt{3} - i$ **26.** $3 + 3i$ **27.** $-4i$

Write the rectangular forms of the following complex numbers (to the nearest tenth).

28. $3 \text{ cis } 35°$ **29.** $3 \text{ cis } 243°$

Write the rectangular forms of the following complex numbers. Leave the result in exact form.

30. $3 \text{ cis } 240°$ **31.** $10 \text{ cis } 330°$

Multiply the following complex numbers.

32. $(2 \text{ cis } 25°)(3 \text{ cis } 45°)$
33. $(2 \text{ cis } 18°)(6.5 \text{ cis } 122°)$

Divide the following complex numbers.

34. $\frac{40 \text{ cis } 120°}{5 \text{ cis } 20°}$ **35.** $\frac{50 \text{ cis } 45°}{100 \text{ cis } 9°}$

36. Compute the cube of $2 \text{ cis } 130°$.
37. Compute the fourth power of $2 \text{ cis } 150°$.
38. Compute an approximation to $(0.8 + 0.6i)^8$ (to the nearest tenth).
39. Find the 4 fourth roots of 16 in exact form.
40. Find the 3 cube roots of -27 in exact form.
41. In electronics one version of Ohm's law says that $I = \frac{V}{Z}$, where I is current, V is voltage, and Z is impedance. Find I in a circuit in which $V = 130 \text{ cis } 25°$ and Z is $30 \text{ cis } 75°$.

[7–2] Graph the following points in polar coordinates.

42. $(2,0)$ **43.** $(2,\pi)$ **44.** $\left(3,\frac{5\pi}{6}\right)$
45. $\left(-6,\frac{11\pi}{6}\right)$ **46.** $\left(1,\frac{\pi}{2}\right)$ **47.** $(-3,\pi)$
48. $(2,1)$ **49.** $(3,4)$ **50.** $(-4,1)$

Convert the following polar coordinates into rectangular coordinates. Leave the result in exact form.

51. $\left(3,\frac{11\pi}{6}\right)$ **52.** $\left(4,\frac{2\pi}{3}\right)$ **53.** $\left(-2,\frac{5\pi}{3}\right)$

Convert the following polar coordinates into rectangular coordinates to two-decimal-place accuracy.

54. $(2,1)$ **55.** $(5,2)$ **56.** $(1,0.5)$

Convert the following rectangular coordinates into polar coordinates to two-decimal-place accuracy.

57. $(2,1)$ **58.** $(-5,3)$ **59.** $(-1,-4)$

Convert the following rectangular equations into polar equations.

60. $y = -3x$ **61.** $y = 4x + 2$
62. $2y^2 - x^2 = 5$ **63.** $y^2 - 3x = 0$

Convert the following polar equations into rectangular equations.

64. $r = \sin \theta$ **65.** $r = 2 \sec \theta$
66. $r^2 = \sin 2\theta$ **67.** $r^2 = \tan \theta$
68. $r \sin \theta = 2$ **69.** $r = \frac{3}{2 - \sin \theta}$

[7–3] Graph the polar equations.

70. $r = -2$ **71.** $\theta = \frac{5\pi}{6}$
72. $2r \sin \theta = 8$ **73.** $r = 3 \cos \theta$
74. $r = \sin 2\theta$ **75.** $r = 3 \cos 3\theta$
76. $r = 1 + 2 \cos \theta$ **77.** $r = 2 - \sin \theta$

Chapter 7 test

Simplify each expression by performing the indicated operations.

1. $(12 - 4i) - (-3 + 2i)$ **2.** $(1 - 4i)(2 + 8i)$

3. $\dfrac{2 - 5i}{2 - 3i}$

4. If $z = 1 + 2i$, evaluate $3z^2 - 2z + 5$.

5. Write the polar form of the number $4 - 5i$ (to the nearest tenth).

6. Write the rectangular form of the number 2 cis 120°. Leave the result in exact form.

7. Multiply (2 cis 325°)(7 cis 145°). Leave the result in polar form.

8. Divide $\dfrac{6 \text{ cis } 140°}{2 \text{ cis } 20°}$. Leave the result in polar form.

9. Compute the cube of 3 cis 150°.

10. Find the 4 fourth roots of -16 in exact form.

Graph the following points in polar coordinates.

11. $\left(2, \dfrac{11\pi}{6}\right)$ **12.** $\left(2, \dfrac{\pi}{3}\right)$ **13.** $(-2, 1)$

Convert the following polar coordinates into rectangular coordinates to two-decimal-place accuracy.

14. $(3, 0.8)$ **15.** $(-5, 2)$

Convert the following rectangular coordinates into polar coordinates. Leave the result in exact form.

16. $(-\sqrt{3}, -1)$ **17.** $(-5, 5)$

Convert the following rectangular equations into polar equations.

18. $y = -3x + 5$ **19.** $2y^2 - x = 5$

Convert the following polar equations into rectangular equations.

20. $r = 2 \csc \theta$ **21.** $r^2 = \cos 2\theta$

Graph the polar equations.

22. $r = -\theta, \theta > 0$ **23.** $r = 2 \csc \theta$

24. $r = 3 \cos 2\theta$